Vortex Lattice Locking in Rotating Two-Component Bose-Einstein Condensates

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The vortex density of a rotating superfluid, divided by its particle mass, dictates the superfluid's angular velocity through the Feynman relation. To find how the Feynman relation applies to superfluid mixtures, we investigate a rotating two-component Bose-Einstein condensate, composed of bosons with different masses. We find that in the case of sufficiently strong interspecies attraction, the vortex lattices of the two condensates lock and rotate at the drive frequency, while the superfluids themselves rotate at two different velocities, whose ratio is the ratio between the particle mass of the two species. In this paper, we characterize the vortex-locked state, establish its regime of stability, and find that it surives within a disk smaller than a critical radius, beyond which vortices become unbound, and the two Bose-gas rings rotate together at the frequency of the external drive.

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After the first experimental realization of Bose-Einstein condensates (BECs) of alkali atoms, their study has experienced enormous advancements [1]. Among the major threads of investigation in BECs has been the study of vortices both experimentally [2, 3, 4] and theoretically [5]. It is known from the classic works of Onsager and Feynman [6, 7] that superfluids rotate by nucleating vortices. When there are several vortices present, they form a triangular Abrikosov vortex lattice [8], with density given by

$$\rho_v = \frac{m\Omega}{\pi\hbar}\,, \tag{1}$$

where m is the mass of a constituent boson and Ω is the rate at which the superfluid – which rotates with the vortex lattice – is being rotated (see, for instance, Ref. [9]). The so-called "Feynman relation" (1) states that, on average, a uniform superfluid rotates like a rigid body. It has been shown that corrections to Eq. (1) due to the typical experimental situation of nonuniform superfluid density resulting from a harmonic trap are small [10] but experimentally observable [11]. Vortex physics becomes much more intriguing in multi-component BEC's. So far, the investigation of vortex lattices in multi-component BECs utilized different hyperfine levels of the constituent atoms to obtain multi-component condensates (e.g., [12, 13]). Thus the mass of all condensate components was identical, and the generalization of Eq. (1) straightforward.

In this paper we ask what are the consequences of the Feynman condition, Eq. (1), in a system of interacting two-component rotating condensates with *different masses*. We find that for sufficiently large attractive interactions, the Feynman condition leads to a novel state. The two components, rather than rotate together at the drive frequency, rotate at angular velocities $\Omega_{1,2}$ inversely proportional to their masses, $m_{1,2}$, such that:

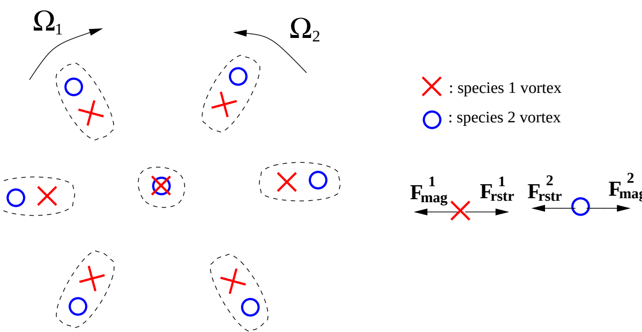

FIG. 1: Schematic diagram of bound vortex pairs (for species with masses m_1 and $m_2 < m_1$) in the rest frame of the vortex lattice in the limit of large interspecies interaction in which the two vortex lattices are locked. The Magnus force $\mathbf{F}_{\rm mag}$ opposes such locking and is balanced by a restoring force $\mathbf{F}_{\rm rstr}$ due to the interspecies vortex-vortex interaction.

$$m_1\Omega_1 = m_2\Omega_2 \tag{2}$$

while the vortex lattices of the two components lock at the drive angular velocity Ω_v, lying between Ω_1 and Ω_2 (see Fig. 1). Qualitatively, the attractive interspecies interaction leads to an attraction between vortices of the two flavors. If it is sufficiently strong, vortices pair, and the lowest-energy vortex configuration then occurs when the vortex lattices of the two flavors are "locked" together, rotate at the same rate, and have essentially the same density. Eq. (1) then reads:

$$\rho_v^1 = \frac{m_1\Omega_1}{\pi\hbar} \approx \frac{m_2\Omega_2}{\pi\hbar} = \rho_v^2 \tag{3}$$

where $\rho_v^{1,2}$ are the vortex densities of the two flavors. This state is strongly related to experiments in Ref. 14, where a vortex lattice was locked to an optical lattice with a similar periodicity.

As we show below, this state survives within a finite disk about the center of the rotating condensate. The relative motion between the vortices and the condensate gives rise to a Magnus force that opposes the interspecies vortex attraction. Beyond a critical distance the Magnus force (Fig. 1) becomes larger then the maximal pairing force, and the vortices become unbound. In this region, the two condensates and their vortices rotate together at the drive frequency Ω_v; the vortex densities in the two flavors are no longer equal, but reflect themselves the mass ratio: $\rho_v^1/m_1 = \rho_v^2/m_2$. Below we derive the characteristics and conditions for the vortex locking state.

Energetics. The energy of weakly interacting Bose-Einstein condensates is well-described by the Gross-Pitaevskii functional for the condensate wave function ψ_α ($\alpha = 1, 2$) for the two atomic species [1]. We consider the situation in which the two condensates are stirred at the same rate Ω_v. Transforming to the rotating frame, our problem becomes time independent. The energy of the two-component system in the rotating frame is given by $E = E_1 + E_2 + E_{12}$, where

$$E_\alpha = \int d^2r \left(\frac{\hbar^2}{2m_\alpha} |\nabla\psi_\alpha|^2 + V_{\text{trap}}(\mathbf{r}) n_\alpha + \frac{g_\alpha}{2} n_\alpha^2 - \hbar\Omega_v \, \psi_\alpha^* \left(-i\frac{\partial}{\partial\varphi}\right) \psi_\alpha \right) \tag{4}$$

describes the energy for each atomic species and g_α is the intraspecies coupling for bosons of flavor α. The interspecies interaction is given by

$$E_{12} = g_{12} \int d^2r [n_1(\mathbf{r}) n_2(\mathbf{r})] \,, \tag{5}$$

where $n_\alpha = |\psi_\alpha|^2$ is the density of flavor α, V_{trap} is the external trapping potential, and the z-component of the angular momentum operator is $L_z = -i\frac{\partial}{\partial\varphi}$ (where φ is the azimuthal coordinate).

The energy E_α of one BEC component is minimized via the nucleation of a vortex lattice rotating with the external drive Ω_v. In the following, we assume that the coherence length $\xi_\alpha = \sqrt{\frac{\hbar^2}{2m_\alpha g_\alpha n_0^\alpha}}$ is much shorter than the characteristic distance between vortices. Each vortex is then well-described by a small core region of size ξ_α, at which the superfluid density drops to zero and its phase field accounts for the flow around the vortex. The vortex core region gives rise to a small constant energy, and we can account for the phase field by writing $\psi_\alpha = \sqrt{n_\alpha} e^{i\theta_\alpha}$, where θ_α determines a lattice of vortices with unit winding number at the positions $\{\mathbf{r}_i^\alpha\}$. We assume that we can write θ_α as a sum of the different vortex contributions $\theta_\alpha = \theta_1^\alpha + \theta_2^\alpha + \ldots$.

With these assumptions, E_α can be written in terms of the positions of the vortices as

$$E_\alpha = \frac{\hbar^2\pi}{m_\alpha} n_0^\alpha \sum_{i\neq j} \log\left(\frac{\xi_\alpha}{|\mathbf{r}_i^\alpha - \mathbf{r}_j^\alpha|} \right) + \hbar\Omega_v n_0^\alpha \pi \sum_i (r_i^\alpha)^2 \,, \tag{6}$$

where we have dropped terms that do not depend on the positions of the vortices and have also neglected effects due to the nonuniform superfluid density [10]. The first term in Eq. (6) is the usual logarithmic interaction between vortices, and the second is the centripetal energy, reflecting the fact that vortices toward the edge of the cloud carry less angular momentum relative to the center of the cloud. In a single-component rotating BEC, the balance of the two terms gives the Feynman condition (1). The equations describe charged particles interacting in two dimensions with a uniform background charge of opposite sign.

The energy E_{12} arising from the interspecies interaction energy is less straightforward to evaluate. Unlike the intraspecies logarithmic interaction, this nonuniversal interaction depends on the details of the short-distance density variations around the vortex cores. For instance, in [15] to study the interaction of a vortex with an optical lattice, a step function having the width of the BEC coherence length was taken. In this work we take a Gaussian depletion around the vortex core:

$$n_\alpha(\mathbf{r}) = n_0^\alpha (1 - e^{-|\mathbf{r}-\mathbf{r}_0|^2/\xi_\alpha^2}) \tag{7}$$

so that the system will be amenable to analytic treatment. For a single vortex this depletion gives the correct behavior at short distances, but not the long distance behavior, in which the density due to a single isolated vortex heals as ξ^2/r^2. As we show in the Appendix, the combined density variations due to the vortex lattice on scales larger than the inter-vortex separation combines only to change the chemical potential (which is proportional to the density correction), by $\frac{\hbar^2}{2m}\pi^2\rho_v^2 r^2$, which just reflects the kinetic energy associated with uniform rotation of the condensate. This can be shown to have a negligible effect on the vortex pairs, which are the focus of this work. Neglecting this piece of the density fluctuation is also consistent with neglecting the intraspecies core-core interactions which is a standard approximation [5]. Indeed, the short distance region is the relevant one for vortex locking; two-species vortex pairs become unbound once their separation is comparable to the coherence lengths, where the approximate form we take for the density profiles is still valid. Evaluating the interaction integral in Eq. (5), and keeping only the contributions due to the interactions between pairs of vortices between different species, gives

$$E_{12} = g_{12} n_0^1 n_0^2 \pi \frac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} \sum_{ij} e^{-\frac{|\mathbf{r}_i^1 - \mathbf{r}_j^2|^2}{\xi_1^2 + \xi_2^2}} \,. \tag{8}$$

We note that to prevent phase separation, the criterion $|g_1 g_2| > |g_{12}|^2$ must be satisfied. Equations (6) and (8) now give the energetics of the system purely in terms of the positions of the vortices.

Forces. An understanding of the locked phases can be obtained by considering the forces acting on the vortices. Equation (6) leads to the well-known Magnus force acting on a vortex of species α [9]:

$$\mathbf{F}^{\alpha}_{\rm mag} = 2\pi\hbar n_0^{\alpha}(\mathbf{v}^{\alpha}_{SF} - \mathbf{v}_v) \times \hat{\kappa}\,, \tag{9}$$

where $\hat{\kappa}$ is a unit vector pointing out of the plane, n_0^{α} is the equilibrium superfluid density for species α (evaluated away from the vortex core), $\mathbf{v}^{\alpha}_{SF} = \frac{\hbar}{m_\alpha}\nabla\theta_\alpha$ is the superfluid velocity for species α (with the vortex on which the force operates excluded from θ_α), and $\mathbf{v}_v$ is the velocity of the vortex. On the other hand, the force arising from the energy in Eq. (8) provides an attractive force between two vortices of different species. It has the form

$$\mathbf{F}^{\alpha}_{\rm rstr} = -2\pi|g_{12}|n_0^1 n_0^2 \frac{\xi_1^2\xi_2^2}{(\xi_1^2+\xi_2^2)^2} e^{-\frac{\mathbf{d}^2}{\xi_1^2+\xi_2^2}}\;\mathbf{d}\,, \tag{10}$$

where $\mathbf{d}$ is the displacement vector between vortices.

Let us first briefly consider the unlocked case where the vortex interspecies interaction force is small. Since the force counteracting the Magnus force is too small, to bind vortex pairs, we must have $\mathbf{F}_{\rm mag} = 0$ for any isolated vortex, which implies that the superfluid velocity must be the same as the vortex velocity. That is, the vortex lattice rotates with the superfluid. Thus, because the two vortex lattices rotate at the same frequency, the two superfluids rotate together at that frequency. Accordingly, for this case, the vortex densities are not equal: $\rho_v^1 = \frac{m_1}{m_2}\rho_v^2$.

We next consider the other extreme, in which the two vortex lattices are locked. Our approach is to consider the forces acting on a bound pair of vortices at distance r from the center of the trap (see Fig. 1). As stated before, the Magnus force for the locked state is nonzero because the superfluids are rotating at different rates. Balancing the forces gives $F^{\alpha}_{\rm mag} = F^{\alpha}_{\rm rstr}$. Because the restoring force acting on either species has the same magnitude, we obtain $F^1_{\rm mag} = F^2_{\rm mag}$. Noting that $v_\alpha = \Omega_\alpha r$ and $v_v = \Omega_v r$ (we are assuming that r is much larger than the distance between the two vortices) gives $n_0^1(\Omega_v - \Omega_1) = n_0^2(\Omega_2 - \Omega_v)$. This, along with the condition $m_1\Omega_1 = m_2\Omega_2$ (from $\rho_v^1 = \rho_v^2$) gives the following relation between the angular velocities:

$$\Omega_1 = \frac{(n_0^1+n_0^2)m_2}{m_1 n_0^2 + m_2 n_0^1}\Omega_v < \Omega_v < \Omega_2 = \frac{(n_0^1+n_0^2)m_1}{m_1 n_0^2 + m_2 n_0^1}\Omega_v\,. \tag{11}$$

Note that unlike the restoring force, the Magnus force grows linearly with distance from the center of the condensate. Thus, at some critical distance r_c, the pairs of vortices invariably become unbound from each other. for radii $r > r_c$ the unlocked phase applies, and after a short healing region, the two condensates rotate at the same frequency. An expression for r_c can be obtained by equating the Magnus force with the maximum possible value for the restoring force:

$$r_c = \sqrt{\frac{1}{2e}}\frac{|g_{12}|}{\hbar\Omega_v}\frac{m_1 n_0^2 + m_2 n_0^1}{m_1 - m_2}\,\frac{\xi_1^2\xi_2^2}{(\xi_1^2+\xi_2^2)^{3/2}}\,. \tag{12}$$

Note that (12) diverges when the masses are equal. In addition, the bound pairs of vortices are pulled further apart at increasing distances from the center of the condensate. The interspecies vortex separation x_v satisfies

$$x_v e^{-\frac{x_v^2}{\xi_1^2+\xi_2^2}} = \frac{r}{r_c\sqrt{2e}}\,\sqrt{\xi_1^2+\xi_2^2}\,, \tag{13}$$

which is valid for $r < r_c$. This introduces a small correction to the vortex density and creates a small shear in the motion of the two condensates.

For the vortex-locked state to be stable up to the critical radius r_c, the superfluid velocities in the rotating frame of the vortex lattice, $|\mathbf{v}^{\alpha}_{SF} - \mathbf{v}_v|$, must not exceed the critical velocity of the superfluid. Otherwise, it would be possible to create elementary excitations from the flow of the superfluid around the vortices. For the two coupled superfluids, the Bogoliubov elementary excitations are given by

$$(\Omega_k)^2 = \frac{1}{2}(E_k^1 + E_k^2) \pm \frac{1}{2}\sqrt{(E_k^1 - E_k^2)^2 + 16 g_{12}^2 n_0^1 n_0^2 \varepsilon_k^1\varepsilon_k^2}\,, \tag{14}$$

where $E_k^{\alpha} = \sqrt{(\varepsilon_k^{\alpha})^2 + 2g_\alpha n_0^{\alpha}\varepsilon_k^{\alpha}}$ and $\varepsilon_k^{\alpha} = \frac{\hbar^2 k^2}{2m_\alpha}$. It is then straightforward to compute the critical velocity $v_c = \min_k(\frac{\Omega_k}{\hbar k})$; one obtains

$$v_c = \min_\alpha\left\{\sqrt{\frac{g_\alpha n_0^{\alpha}}{m_\alpha}}\right\} = \min_\alpha\left\{\frac{\hbar}{\sqrt{2}m_\alpha\xi_\alpha}\right\}\,. \tag{15}$$

The superfluid velocity of species 1 or 2 in the vortex lattice frame evaluated at r_c (where it is maximal) is given by

$$|\mathbf{v}_{SF}^{\{1,2\}} - \mathbf{v}_v| = \frac{1}{\sqrt{2e}}\frac{|g_{12}|}{g_{\{2,1\}}}\frac{\hbar}{2m_{\{2,1\}}}\frac{\xi^2_{\{1,2\}}}{(\xi_1^2+\xi_2^2)^{3/2}}\,. \tag{16}$$

The condition $|\mathbf{v}^{\alpha}_{SF} - \mathbf{v}_v| < v_c$ must be checked so that the vortex-locked state is stable against the creation of elementary excitations. For instance, it can be shown that the system is stable against creating such elementary excitations if the conditions $\frac{1}{10} \le \frac{n_1^0}{n_2^0} \le 10$ and $\frac{1}{10} \le \frac{m_1}{m_2} \le 10$ are satisfied.

Numerical Simulations. Now that the expected types of phases have been discussed, we minimize the total energy $E = E_1 + E_2 + E_{12}$ as a function of the vortex positions given by Eqs. (6) and (8). The ability to compute analytical expressions for the gradients of the energy

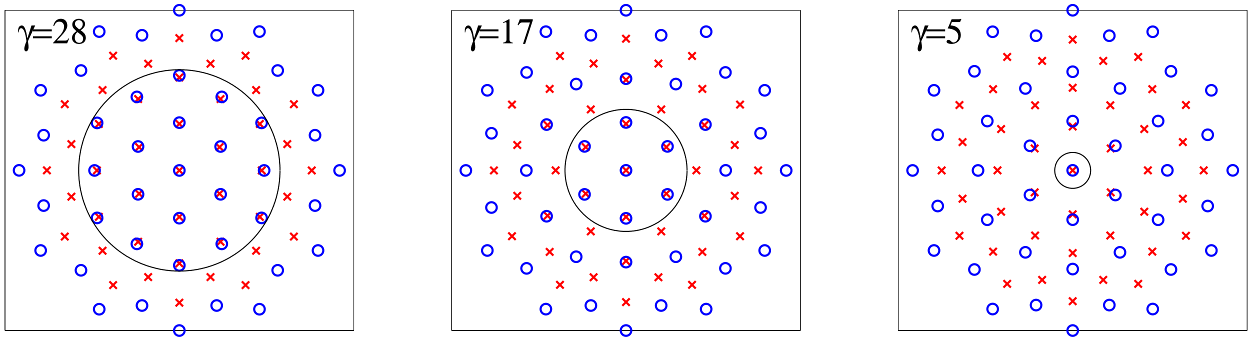

FIG. 2: Vortex lattice for two-component condensate (with 43 vortices of each species) for different values of $\gamma = |g_{ab}|n_0/(\hbar\Omega_v)$. Circles are shown for the theoretical prediction for the critical radius Eq. (19) at which the vortices become unbound.

as a function of the vortex positions $\nabla_{\{\mathbf{r}_i\}}E$ allow us to apply the steepest descent method for the minimization. Specifically, starting with an initial configuration for the vortex positions $\{\mathbf{r}_i^{(0)}\}$, we perform the one-dimensional minimization of

$$E\left(\{\mathbf{r}_i^{(n)}\} - \lambda\nabla_{\{\mathbf{r}_i\}}E\right) \tag{17}$$

over λ, where $\nabla_{\{\mathbf{r}_i\}}E$ is evaluated at $\{\mathbf{r}_i^n\}$. The new vortex positions are given by

$$\{\mathbf{r}_i^{(n+1)}\} = \{\mathbf{r}_i^{(n)}\} - \lambda_{\min}\nabla_{\{\mathbf{r}_i\}}E\,, \tag{18}$$

and the above procedure is repeated until it converges.

To simplify the analysis, we restrict our attention to the case in which the equilibrium densities and healing lengths of the two condensate components are equal: $n_0 \equiv n_0^1 = n_0^2$, $\xi \equiv \xi_1 = \xi_2$. Motivated by the example of a ^{133}Cs-^{87}Rb condensate [16], we fix the mass ratio to be $m_1/m_2 = 1.5$. The vortex interaction strength is parametrized by $\gamma = \frac{|g_{12}|n_0}{\hbar\Omega_v}$, which we vary while keeping the quantities $\frac{\hbar\pi}{\Omega_v m_\alpha}\frac{1}{\pi\xi^2}$ for $\alpha = 1, 2$ fixed. (The total energy has been scaled by $\hbar\Omega_v\pi n_0\xi^2$.) We set the ratio of the "average" vortex lattice constant to the coherence length $a_{\rm lat}/\xi$=10 (which is consistent with typical experiments). We define $a_{\rm lat}$ by $\frac{2}{\sqrt{3}a_{\rm lat}^2} = \frac{\tilde{m}\Omega_v}{\pi\hbar}$, where $\tilde{m} = \frac{2m_1m_2}{m_1+m_2}$, and consider a system with 43 vortices of each species. The results for such a calculation are shown in Fig. 2. We also plot our prediction for the critical radius at which the vortex pairs become unbound [see Eq. (12)], which for equal densities is

$$r_c = \frac{\gamma}{4\sqrt{e}}\frac{m_1+m_2}{m_1-m_2}\,\xi. \tag{19}$$

This prediction agrees quite well with our numerical results.

Experimental realization. A very promising candidate for the realization of these locked states is a ^{133}Cs-^{87}Rb condensate mixture [16], which has a mass ratio of about 1.5. One has exquisite control over the self-scattering length of Cesium [17], and a Cs-Rb mixture is also expected to exhibit interspecies Feshbach resonances. This allows one to control the interaction g_{12} over a wide range; such interspecies resonances have recently been identified for Li-Na [18] and Rb-K [19] mixtures.

Conclusions. In this paper, we described a novel state of rotating interacting condensates with unequal masses in the Thomas-Fermi regime. The possibility of locking the two individual vortex lattices yields a remarkable demonstration of the nonintuitive behavior of superfluids: the two gasses, rather than equilibrating to the same speed, prefer to move at different angular velocities that are inversely proportional to their masses. The vortex-locking of the different-mass condensates is also an example of synchronization: a phenomenon that is ubiquitous in physics, biology, and other fields [20]. Already in single-mass mixtures, a rich variety of vortex dynamics arises from the extra degrees-of-freedom, resulting in such effects as the formation of square vortex lattice, as well as topologically nontrivial defects such as skyrmions or hedgehogs [13, 21]. To investigate these effects in the different-mass mixtures, as well as to better establish the locked state we proposed in this manuscript, this system must be numerically investigated by solving the appropriate dynamical Gross-Pitaevskii equations. Such numerical investigation will also allow one to find the preferred lattice geometry of the locked vortex-lattice. Other directions for future study involve dynamical aspects such as Tkachenko modes [22]) of the locked state, as well as whether a similar state survives in the Landau regime of large vortex density.

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APPENDIX

In this appendix we discuss the change in the superfluid density as a result of a vortex lattice, and its effect on the validity of approximation (7) and the resulting expression for the interspecies vortex attraction, Eq. (8) and (5).

First consider a single species which has the energy functional

$$E = \int d^2r \left(\frac{\hbar^2}{2m} |\nabla\psi|^2 + V_{\rm trap}(r)|\psi|^2 + \frac{1}{2} g |\psi|^4 \right). \quad (20)$$

We write $\psi = f e^{i\theta}$ where f is real and θ contains information about the positions of the vortices as $\theta = \sum_i \theta_i$. By varying f we obtain the equation determining the superfluid density which minimizes E

$$-\frac{\hbar^2}{2m}\nabla^2 f + \frac{\hbar^2}{2m}|\nabla\theta|^2 f + V_{\rm trap} f + g f^3 = \mu f \quad (21)$$

for the particular vortex configuration. First let us consider a single vortex taken to be at the origin $\nabla\theta = \hat{z} \times \frac{\hat{r}}{r}$. The long-distance behavior $r \gg \xi$ is obtained from the Thomas-Fermi approximation (neglecting the $\nabla^2 f$ in Eq. 21) and we obtain

$$f \approx \frac{\mu}{g}\left(1 - \frac{\hbar^2}{2m\mu r^2}\right) = n_0 \left(1 - \frac{\xi^2}{r^2}\right) \quad (22)$$

where we have neglected the contribution from the trapping potential. This implies that the suppression of the density is due to the kinetic energy in the supercurrent, which counters the condensation energy of the BEC.

Eq. (22) seems to imply that at large distances from a vortex core, the interspecies vortex-vortex interaction will include a persistent power-law component, and die off only as $1/r^2$, rather than as an exponential. But the observation that the power law decay reflects the current-induced superfluid suppression, allows as to ignore the power-law decay in a many-vortex situation, with the argument as follows. Let us consider a vortex lattice, and evaluate f at a position which is several coherence lengths away from any vortex. This allows us to invoke the continuum approximation and write

$$\nabla\theta(\mathbf{r}) = \hat{\mathbf{z}} \times \sum_{\mathbf{i}} \frac{\mathbf{r} - \mathbf{r_i}}{|\mathbf{r} - \mathbf{r_i}|^2} = \pi\rho_{\mathbf{v}} \hat{\mathbf{z}} \times \mathbf{r} \quad (23)$$

where ρ_v is the density of the vortices. Each individual contribution alone in this sum would yield a $\propto \frac{1}{r^2}$ dependence in the density corrections, but the vector sum of the velocities of all vortices squared is not a simple sum of the $1/r^2$ corrections. Inserting this into the equation for the density profile one finds:

$$-\frac{\hbar^2}{2m}\nabla^2 f + \frac{\hbar^2}{2m}\pi^2\rho_v^2 r^2 f + V_{\rm trap} f + g f^3 = \mu f. \quad (24)$$

Thus the combined vortex effect re-normalizes the trapping potential and does not need to be explicitly taken into account. To get an estimate for the magnitude of such a renormalization, one can compare this term with the chemical potential

$$\frac{\frac{\hbar^2}{2m}\pi^2\rho_v^2 r^2}{\mu} \sim \left(\frac{\xi}{a_{lat}}\frac{r}{a_{lat}}\right)^2 \quad (25)$$

where a_{lat} is the vortex lattice constant which is small for typical experiments.

For a two-component BEC, the situation is similar for vortices which are many coherence lengths away from each other. On the other hand, when the cores of the different types of overlap, their interaction needs to be explicitly calculated, and the continuum approximation cannot be used. This is the case for paired-vortex configurations. Since the combined effect of far-away vortices on a locked pair is small, the locking depends only on the short distance density profile taken. Had we used the step-function potential interaction between two vortices of Ref. 15 the results would only differ from the Gaussian depletion Eq. 7 by small quantitative amounts.

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